EXERCISES [MAI 3.14]

CROSS PRODUCT

SOLUTIONS

Compiled by: Christos Nikolaidis

A. Paper 1 questions (SHORT)

1. (a) $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}=2+0+9=11$.

(a) $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}-6\\-3\\4\end{pmatrix}$ and $\boldsymbol{b}\times\boldsymbol{a}=\begin{pmatrix}6\\3\\-4\end{pmatrix}$

(b) $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}=-12+0+12=0$, $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}=-6-6+12=0$

2. (a) $(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=11\begin{pmatrix}4\\1\\0\end{pmatrix}=\begin{pmatrix}44\\11\\0\end{pmatrix}$

(b) $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{pmatrix}-6\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}4\\1\\0\end{pmatrix}=-24-3=-27$, $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{pmatrix}2\\0\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\12\\-7\end{pmatrix}-6-21=-27$

(c) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\begin{pmatrix}-6\\-3\\4\end{pmatrix}\times\begin{pmatrix}4\\1\\0\end{pmatrix}=\begin{pmatrix}-4\\16\\6\end{pmatrix}$

3. (a) $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}2\\-2\\2\end{pmatrix}\times\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}-6\\2\\8\end{pmatrix}$

(b) Area of triangle ABC = $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{\sqrt{36+4+64}}{2}=\frac{\sqrt{104}}{2}=\sqrt{26}$

(c) Volume of tetrahedron ABCD $=\frac{1}{6}|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})|=\frac{1}{6}\left|\begin{pmatrix}4\\5\\0\end{pmatrix}\cdot\begin{pmatrix}-6\\2\\8\end{pmatrix}\right|=\frac{7}{3}$

4. (a) The vector product, $\boldsymbol{p}\times\boldsymbol{q}=\begin{pmatrix}-7\\7\\7\end{pmatrix}$ $(=-7\boldsymbol{i}+7\boldsymbol{j}+7\boldsymbol{k})$

(b) Area of parallelogram = $|p\times q|=\sqrt{147}$ or $7\sqrt{3}$ or 12.1 units^2

5. (a) A perpendicular vector can be found from the vector product

$\overrightarrow{OP}\times\overrightarrow{OQ}=\vec{i}-3\vec{j}-5\vec{k}$

(b) Area $\Delta OPQ=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|=\frac{\sqrt{35}}{2}$

6. (a) $\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}-1\\p\\6\end{pmatrix} = \begin{pmatrix}6\\-12\\2p+1\end{pmatrix}$

(b) $\begin{pmatrix}6\\-12\\2p+1\end{pmatrix}$ parallel to $\begin{pmatrix}2\\-4\\3\end{pmatrix} \Rightarrow \frac{6}{2}=\frac{-12}{-4}=\frac{2p+1}{3}$

$\Rightarrow \frac{2p+1}{3} = 3 \Rightarrow 2p+1 = 9 \Rightarrow p = 4$

7. $\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}-3\\2\\2\end{pmatrix} = \begin{pmatrix}6\\1\\8\end{pmatrix}$

$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \begin{pmatrix}6\\1\\8\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\4\end{pmatrix} = 12 - 3 + 32 = 41$

8. (a) $\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}-1-2\\-4+2\\-2-2\end{pmatrix} = \begin{pmatrix}-3\\-2\\-4\end{pmatrix} = -3\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

(b) LHS $= (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \begin{pmatrix}-3\\-2\\-4\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}-4+8\\-4+6\\-6+2\end{pmatrix} = \begin{pmatrix}4\\2\\-4\end{pmatrix}$

$\boldsymbol{b}\cdot\boldsymbol{c} = 2 - 2 - 2 = -2$

RHS $= -(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} = 2\boldsymbol{a} = 2\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}4\\2\\-4\end{pmatrix}$

9. Such a vector is the cross product $\boldsymbol{d}_1\times\boldsymbol{d}_2$, where $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ are the direction vectors of the two lines

$\boldsymbol{d}_1\times\boldsymbol{d}_2 = \begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}3+2\\0-6\\2-0\end{pmatrix} = \begin{pmatrix}5\\-6\\2\end{pmatrix}$ (or any multiple)

10.

Two bounding vectors are $\vec{AB} = \begin{pmatrix}7\\-1\\-1\end{pmatrix} - \begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}4\\-3\\-1\end{pmatrix}$ and $\vec{AD} = \begin{pmatrix}6\\0\\1\end{pmatrix} - \begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}3\\-2\\1\end{pmatrix}$

$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{pmatrix}4\\-3\\-1\end{pmatrix}\times\begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}-5\\-7\\1\end{pmatrix}$

Area $= \sqrt{25+49+1} = \sqrt{75}$ $(= 5\sqrt{3})$

11. $\overrightarrow{BA} = \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} 5 \\ 0 \\ -12 \end{pmatrix}$

$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{pmatrix} 36 \\ -20 \\ 15 \end{pmatrix}$

Area = $\frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2}\sqrt{36^2 + 20^2 + 15^2} = \frac{1}{2}\sqrt{1921} = 21.9$

12. $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

Area = $\frac{1}{2}\sqrt{4+9+36} = \frac{7}{2}$

13. **METHOD 1**

Use of $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$

$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 (1 - \cos^2\theta)$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \cos^2\theta$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta)^2$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2$

Hence LHS = RHS

METHOD 2

Use of $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$

$|\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta)^2$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \cos^2\theta$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 (1 - \cos^2\theta)$

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta = |\boldsymbol{a} \times \boldsymbol{b}|^2$

Hence LHS = RHS

14. (a) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \Rightarrow \text{P} = (4, 0, -3) \qquad \overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OC} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \text{Q} = (3, 3, 0)$

$\overrightarrow{OR} = \overrightarrow{OB} + \overrightarrow{OC} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{R} = (3, 1, 1) \qquad \overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OC} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \text{S} = (5, 2, -1)$

(b) $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix}, \quad \overrightarrow{OA} \times \overrightarrow{OB} \bullet \overrightarrow{OC} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -3 - 4 - 8 = -15$

$\boldsymbol{V} = 15$

B. Paper 2 questions (LONG)

15. (a) area of parallelogram= $|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{36+9+16} = \sqrt{61}$, area of triangle= $\dfrac{\sqrt{61}}{2}$

(b) $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = -27$

(c) (i) volume of parallelepiped $|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}| = 27$

(d) the volume of tetrahedron $\dfrac{1}{6}|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}| = \dfrac{27}{6}$

(e) For the tetrahedron

$Volume = \dfrac{1}{3}(base)\times(height) \Leftrightarrow \dfrac{27}{6} = \dfrac{1}{3}\dfrac{\sqrt{61}}{2}h \Leftrightarrow h = \dfrac{27}{\sqrt{61}}$

(we may also use: *Volume of parallelepiped* = (*base*) × (*height*))

16. (a) P(4, 1, –1), Q(3, 3, 5), R (1, 0, 2*c*),

$\overrightarrow{QR} = \begin{pmatrix}-2\\-3\\2c-5\end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix}-3\\-1\\2c+1\end{pmatrix}$

$\overrightarrow{QR}$ is perpendicular to $\overrightarrow{PR} \Leftrightarrow \overrightarrow{QR}\cdot\overrightarrow{PR} = 0 \Leftrightarrow 6 + 3 + (2c-5)(2c+1) = 0$

$\Rightarrow 4c^2 - 8c + 4 = 0 \Rightarrow (c-1)^2 = 0 \Rightarrow c = 1$

(b) $\overrightarrow{PS} = \begin{pmatrix}-3\\0\\3\end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix}-3\\-1\\3\end{pmatrix} \quad \overrightarrow{PS}\times\overrightarrow{PR} = \begin{pmatrix}3\\0\\3\end{pmatrix}$

(c) $\vec{r} = \begin{pmatrix}3\\3\\5\end{pmatrix} + t\begin{pmatrix}-3\\-1\\3\end{pmatrix}$

17. (a) $\overrightarrow{AB} = \begin{pmatrix}-1\\-3\\1\end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

(b) $\overrightarrow{AB}\times\overrightarrow{BC} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

(c) Area of ΔABC = $\dfrac{1}{2}\sqrt{1+1+4} = \dfrac{\sqrt{6}}{2}$

(d) $\vec{r} = \begin{pmatrix}2\\-1\\-6\end{pmatrix} + t\begin{pmatrix}-1\\1\\2\end{pmatrix}$, thus

$x = 2 - t$
$y = -1 + t$
$z = -6 + 2t$

18. (a) (i) $\overrightarrow{AB} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = -5\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$

(ii) Area $= \frac{1}{2} \times \left| -5\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} \right| = \frac{\sqrt{35}}{2}$ $(\cong 2.96)$

(b) L_1: $\vec{\boldsymbol{r}} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$

L_2: $\vec{\boldsymbol{r}} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix}$

L_3: $\vec{\boldsymbol{r}} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$